

General Relativistic Shift in Plasma Frequency due to a Slowly Rotating Compact Star

Babur M. Mirza

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Abstract We investigate the effects of a slowly rotating compact gravitational source on plasma oscillations using the gravitoelectromagnetic approximation to the general theory of relativity. It is shown that there is a shift in the plasma frequency and hence in the refractive index of the plasma due to the gravitomagnetic force. Estimates for the difference in frequency of radially transmitted electromagnetic signals are given for typical compact star candidates. The proposed shift provides a new test of general theory of relativity in the slow rotation approximation.

1 Introduction

Matter surrounding compact gravitational sources (classed as neutron stars, pulsars, and white dwarfs) exists in a highly ionized state plasma. The ionization occurs largely due to the increase in the mean collisional rate of the atoms constituting the star's atmosphere [1]. Also a compact star itself is formed of a plasma in a degenerate state which close to the surface of the star forms the so called ion crust consisting of a high density ion plasma. Outside the ion crust the co-rotating plasma is rather dilute and highly conducting [2]. Collectively a plasma, both in a degenerate form and as a dilute and highly conducting medium, behaves as an oscillating system having a characteristic (resonant) frequency called the plasma frequency. In particular the refractive index of a plasma medium is determined by the plasma frequency, which in turn determines the transmission and reflection of electromagnetic radiation for the plasma [3]. Whereas under normal physical situations where weak gravitational field is involved, such as for main sequence stars, it is common to extrapolate theories well tested in terrestrial laboratories, such as the Newtonian theory of gravitation. However when gravitational field is strong (e.g., as for compact stars) the Newtonian approximation cannot be legitimately used. In such cases general relativistic effects become important for an adequate description of the phenomenon. Usually these effects are not directly observable but are manifest in an indirect way, for example via interaction with the magnetic field, in the

B.M. Mirza (✉)
Department of Mathematics, Quaid-i-Azam University, Islamabad 45320, Pakistan
e-mail: bmmirza2002@yahoo.com

accretion of matter, and other material and radiative processes occurring in vicinity of the star. An investigation of these effects is not only of interest for astrophysical processes but is also important for testing the general theory of relativity [4].

In this paper we study the effects of a rotating compact gravitational source on the plasma oscillations using gravitoelectromagnetic approximation to the general theory of relativity. Based on these investigations it is proposed that the characteristic frequency and consequently the refractive index of the plasma undergoes a shift due to the gravitomagnetic force. The shift depends on the intrinsic parameters of the gravitational source such as its mass, angular speed, and also on component of the angular frequency vector of the star to the plane of oscillation of the plasma. We estimate the effects of the variation in the refractive index of the plasma medium on electromagnetic waves propagating radially through the plasma around the star, and discuss their relevance to observation. Throughout we use the gravitational units where $G = 1 = c$ unless mentioned otherwise.

2 The Gravitoelectromagnetic Approximation

The field equations of general theory of relativity, for a slowly rotating gravitational source and sufficiently weak gravitational field, bear a remarkable formal similarity with the fundamental equations of classical electromagnetism [5, 6]. For example in Einstein's theory of gravitation the trajectory of a test particle in vicinity of a gravitational source is given by the geodesic; assuming, to the first order of approximation, the star to be a slowly rotating sphere of mass M and radius R , then according to the geodesic equation, a test particle of mass m , at a radial distance r , experiences a force (analogous to the Lorentz force law in classical electromagnetism) given by

$$\mathbf{F}_g = m(\mathbf{G} + \mathbf{v} \times \mathbf{H}), \quad (1)$$

where

$$\mathbf{G} = -M\hat{\mathbf{r}}/r^2, \quad (2)$$

is the gravitoelectric force given by the Newtonian gravitational force per unit mass, and $\hat{\mathbf{r}}$ is a unit vector in the radial direction. The term $m(\mathbf{v} \times \mathbf{H})$ is the gravitomagnetic force where \mathbf{H} is given by

$$\mathbf{H} = -\frac{12}{5}MR^2\left(\Omega \cdot \mathbf{r} \frac{\mathbf{r}}{r^5} - \frac{1}{3} \frac{\Omega}{r^3}\right), \quad (3)$$

Ω being the angular velocity of the gravitational source, and \mathbf{v} is velocity of the test particle. The approximation, so-called gravitoelectromagnetic approximation to the general theory of relativity, is generally valid for most astrophysical situations where general relativistic effects are important, such as for compact stars. The physical significance of the gravitomagnetic potential has been demonstrated [7] by showing the independence of the potential from a particular frame and particular coordinate system. Physically it can be interpreted as 'gravitomagnetic current' induced in the vicinity of the gravitational source due to its rotation.

3 Estimate for the Plasma Frequency Shift

For convenience let the orientation of the coordinates triplet (x, y, z) be such that the angular velocity vector $\mathbf{\Omega}$ is along the positive z -axis. Since the plasma oscillations occur within a very small region of space we have, for the components of acceleration due to gravitoelectric force, \mathbf{G} to be of constant magnitude g along each direction (x, y, z) . In this case the components of \mathbf{H} are $(0, 0, H)$, therefore effective components of the gravitomagnetic force lie in the xy -plane. We take, without loss of generality, the xy -plane to be the equatorial plane of the star [8, 9].

The equations of motion for the plasma oscillations can thus be written as

$$\frac{d^2x}{dt^2} = -\omega_p^2 x - g - H \frac{dy}{dt}, \tag{4}$$

$$\frac{d^2y}{dt^2} = -\omega_p^2 y - g + H \frac{dx}{dt}, \tag{5}$$

where $\omega_p = \sqrt{N_0 e^2 / m \epsilon_0}$ is the Newtonian plasma frequency, N_0 being the electron number density of the plasma per centimeter, e is the electronic or ionic charge, and ϵ_0 is the dielectric constant. It is clear from (4) and (5) that H has dimensions cycle per unit time i.e. of frequency.

To investigate the resonance frequency for the above system of equations, we assume that the solutions to (4), and (5) can be expressed as $x = a \exp(i\omega t) - g/\omega_p^2$ and $y = b \exp(i\omega t) - g/\omega_p^2$ where ω is the applied external frequency. The system can then be written as a single matrix equation

$$\begin{bmatrix} \omega^2 - \omega_p^2 & -i\omega H \\ i\omega H & \omega^2 - \omega_p^2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = 0. \tag{6}$$

For solution to exist matrix equation (6) implies that,

$$\begin{vmatrix} \omega^2 - \omega_p^2 & -i\omega H \\ i\omega H & \omega^2 - \omega_p^2 \end{vmatrix} \equiv 0. \tag{7}$$

Since $H \ll \omega_p$, we expand the determinant and neglect terms involving squares and higher powers of H/ω_p . This gives the following expression for resonant frequency of the oscillating plasma

$$\omega \simeq \omega_p - H/2. \tag{8}$$

Here we notice that the shift depends on the intrinsic characteristics of the gravitational source such as its mass and angular speed; it is therefore of significance for any physical phenomenon related to the plasma frequency provided that the gravitational source is sufficiently dense and has high a rotational frequency. As we shall notice in the following that compact astrophysical objects with density reaching up to 10^{28} kg/m^3 and rotational frequency 1 kHz are likely candidates for gravitomagnetic effects to be detectable.

Further assuming χ to be the angle between the radius vector \mathbf{r} (lying in the plane of oscillations of the plasma) and the angular momentum vector $\mathbf{\Omega}$, it follows from expression (3) that close to the surface of the star, i.e. at $r \simeq R$, the gravitomagnetic force has magnitude given by

$$H \equiv |\mathbf{H}| \simeq \mu \sqrt{1 + 3 \cos^2 \chi}, \tag{9}$$

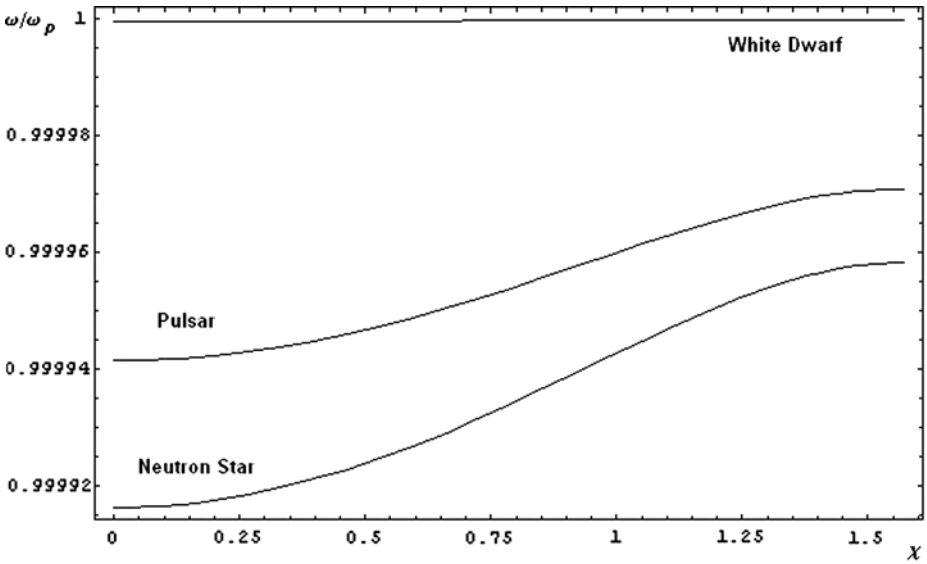


Fig. 1 Plots for the shift ω/ω_p in the plasma frequency of a compact star atmosphere as a function of the angle of inclination χ of the plane of observation to the angular frequency vector for the case of a white dwarf ($M = 1M_{\odot} = 1.989 \times 10^{30}$ kg, $R = 7 \times 10^6$ m, $\Omega = 1$ Hz, $\omega_p = 5.65 \times 10^2$ Hz), a pulsar ($M = 1.4M_{\odot}$, $R = 3 \times 10^4$ m, $\Omega = 30$ Hz, $\omega_p = 5.65 \times 10^4$ Hz), and a neutron star ($M = 2M_{\odot}$, $R = 1 \times 10^4$ m, $\Omega = 1$ kHz, $\omega_p = 5.65 \times 10^6$ Hz)

where $\mu = (4GM/5Rc^2)\Omega$, Ω being the magnitude of the angular velocity vector.

Substituting from expression (9) into (8) we obtain an expression relating plasma frequency ω_p to the angle χ :

$$\frac{\omega}{\omega_p} \simeq 1 - \frac{\mu}{2\omega_p} \sqrt{1 + 3 \cos^2 \chi}, \quad \mu \ll \omega_p. \tag{10}$$

Here the requirements $H \ll \omega_p$ and $\mu \ll \omega_p$ are generally valid for typical compact stellar sources with μ ranging from 0.1687×10^{-3} Hz (for a typical white dwarf of mass $1M_{\odot} = 1.989 \times 10^{30}$ kg, radius 7×10^6 m and angular frequency 1 Hz) to 236.2932 Hz (for a typical neutron star of mass $2M_{\odot}$, radius 1×10^4 m, and angular frequency 1 kHz) with corresponding plasma frequency ranging approximately from 10^3 to 10^6 Hz or above.

In expression (10) we notice that the shift in plasma frequency depends not only on the mass, radius and angular frequency of the compact star, via the parameter μ , but also on the angle of inclination χ . A plot between the plasma frequency shift and the angle χ ; based on expression (10) for the cases of neutron star, pulsar, and white dwarf for the typical values of mass, radius and angular frequency, is given in Fig. 1.

4 Refractive Index of the Plasma

To study the effects of plasma frequency shift on the refractive index of the plasma medium, we denote $\omega_p^{shifted} \equiv \omega_p - H/2$, where H is given by (9). Then in terms of plasma frequency we have the following expression for the index of refraction ϵ of the medium inside the

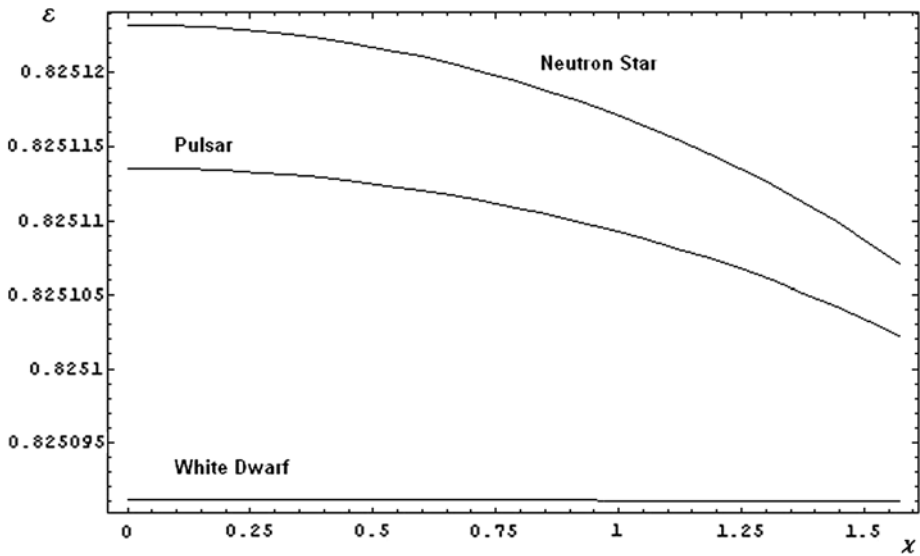


Fig. 2 Plots for the refractive index ϵ of a compact star atmosphere as a function of the angle of inclination χ of the plane of observation to the angular frequency vector for the case of a white dwarf ($M = 1M_{\odot} = 1.989 \times 10^{30}$ kg, $R = 7 \times 10^6$ m, $\Omega = 1$ Hz, $\omega_p = 5.65 \times 10^2$ Hz), a pulsar ($M = 1.4M_{\odot}$, $R = 3 \times 10^4$ m, $\Omega = 30$ Hz, $\omega_p = 5.65 \times 10^4$ Hz), and a neutron star ($M = 2M_{\odot}$, $R = 1 \times 10^4$ m, $\Omega = 1$ kHz, $\omega_p = 5.65 \times 10^6$ Hz) for the propagation of an electromagnetic signal of angular frequency 10^3 , 10^5 , and 10^7 Hz respectively

plasma

$$\epsilon = \sqrt{1 - \left(\frac{\omega_p^{shifted}}{\omega_{EM}}\right)^2}, \tag{11}$$

where ω_{EM} is the angular frequency of an electromagnetic signal propagating through the plasma. From (11) we note that the index of refraction inside the plasma depends, apart from ω_{EM} i.e., the signal frequency, on four factors; namely the three parameters mass M , radius R , and the rotational frequency Ω (through μ) related to the star, and the angle χ related to the plane of oscillations of the plasma (with direction of Ω being fixed). For a given compact source the index depends only on χ when the electromagnetic signal is of some given angular frequency. We plot this dependence of the index ϵ on the angle χ in Fig. 2. We notice that the plasma medium becomes rarer as χ varies from 0 to $\pi/2$. Therefore a radially out-going electromagnetic signal, of given frequency, losses lesser energy in the equatorial plane than in the plane orthogonal to it, namely the plane in which \mathbf{H} lies. This difference in the energy can be estimated in terms of the frequency of electromagnetic waves radially escaping the star as follows.

Let $\delta\omega_{EM}$ be the angular frequency range of the radially out-going waves, then from the usual conditions propagation of an electromagnetic signal in continuous media we have for the transmission and reflection conditions:

$$\delta\omega_{EM} \leq \omega_p^{shifted}, \quad \text{reflection;} \tag{12}$$

$$\delta\omega_{EM} > \omega_p^{shifted}, \quad \text{transmission.} \tag{13}$$

Now since the shift in plasma frequency in the equatorial plane ($\chi = \pi/2$) is $\omega_p - \mu/2$ whereas in the plane orthogonal to it ($\chi = 0$) the shift is $\omega_p - \mu$. Denoting the frequency shift in the equatorial plane by $\omega_{p\parallel}^{shifted}$ and by $\omega_{p\perp}^{shifted}$ for the orthogonal plane, we find that the maximum difference in the frequencies transmitted through the plasma atmosphere is given by

$$|\omega_{p\perp}^{shifted} - \omega_{p\parallel}^{shifted}|_{\max} = \frac{\mu}{2}, \quad (14)$$

μ being dependent on the mass, radius, and angular frequency of the compact star.

It is worth emphasizing here that even if the plasma co-rotates with the star (for instance close to the ion crust) still the gravitomagnetic force will effect the oscillations of the plasma and hence the refractive index, for plasma oscillations and the gravitomagnetic force both are along the radial direction.

5 Conclusions

In view of the above investigations we arrive at the following conclusions:

1. there is a shift in the characteristic frequency and hence in the refractive index of the plasma surrounding the star;
2. the shift not only depends on the intrinsic parameters of the star, i.e., its mass and radius, but is also dependent on the component of the angular frequency vector of the star to the plane of oscillation of the plasma;
3. for an electromagnetic wave of given frequency the shift in the refractive index results in changing the allowed range of frequencies transmitted through the star's atmosphere, by a maximum amount $\mu/2$ (with μ as defined for expression (9) above).

Although the predicted difference in the frequency of electromagnetic radiation emitted by a typical white dwarf (μ being about 0.1687×10^{-3} Hz) is small, for a typical pulsar ($\mu \simeq 1.6540$ Hz) and especially for a neutron star ($\mu \simeq 236.2932$ Hz) the shift may possibly be observed (for example in radiation spectra of various compact sources), when the effects of various contingencies (such as dispersion due to interstellar gases, effects due to the atmosphere of the Earth, etc.) are isolated.

References

1. Rose, W.K.: *Advanced Stellar Astrophysics*. Cambridge University Press, New York (1998)
2. Glendenning, N.K.: *Compact Stars*. Springer, New York (1997)
3. Dougherty, J.P.: In: Dendy, R. (ed.) *Plasma Physics*, p. 58. Cambridge University Press, New York (1993)
4. Jantzen, R.T., Mac Keiser, G. (eds.): *The Seventh Marcel Grossmann Meeting on General Relativity*. Part A, pp. 519–529. World Scientific, New Jersey (1996)
5. Ruggiero, M.L., Tartaglia, A.: *Nuovo Cim. B* **117**, 743 (2002)
6. Rindler, W.: *Phys. Lett. A* **233**, 25 (1997)
7. Ciufolini, I.: *Class. Quantum Gravity* **11**, A73 (1994)
8. Ahmedov, B.J., Rakhmatov, N.I.: *Found. Phys.* **33**, 625 (2003)
9. Mirza, B.M.: *Int. J. Mod. Phys D* **13**, 327 (2004)